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Solution of the Vlasov equation for a static self-consistent potential

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Abstract. A generating function formalism for solving the Vlasov equation for a static self-consistent potential is presented. The transition density, transition current density and linear response function are derived. As an example the response to $I^\pi = 2^+$ and $I^\pi = 2^-$ fields is studied for the case where the self-consistent potential is a harmonic oscillator.

1. Introduction

A classical many-body approach to the independent-particle approximation gives much insight into the interpretation of the properties of the collective excitations in nuclei. For this reason the Vlasov equation, the classical limit of the time-dependent Hartree-Fock (TDHF) equation, has been used recently as the starting point for several semi-classical models which aim to understand these properties [1-5].

In the present work we introduce a mathematical formalism for solving the Vlasov equation for small vibrations around a spherical equilibrium shape in a fixed self-consistent potential.

2. Solution of the Vlasov equation

The Vlasov equation describes the time evolution of the distribution function $f(\boldsymbol{r}, \boldsymbol{p}, t)$ in phase space in the independent-particle approximation

$$\partial f/\partial t + \{f, h_0\} = 0 \tag{2.1}$$

where h_0 is the single-particle Hamiltonian

$$h_0 = p^2/2m + U(r) \tag{2.2}$$

$\{\ ,\ \}$ are Poisson brackets, and $U(r)$ is a fixed self-consistent interaction.

Let $f_0(\boldsymbol{r}, \boldsymbol{p})$ be the distribution function that describes a stationary state, i.e.

$$\{f_0, h_0\} = 0.$$

We introduce the set of all the distribution functions $f(\boldsymbol{r}, \boldsymbol{p})$ which are obtained from f_0 by a canonical transformation [3]

$$\Lambda = \{f(\boldsymbol{r}, \boldsymbol{p}): f(\boldsymbol{r}, \boldsymbol{p}) = f_0(\boldsymbol{r}, \boldsymbol{p}) + \{f_0, F\}(\boldsymbol{r}, \boldsymbol{p}) + \tfrac{1}{2}\{\{f_0, F\}, F\}(\boldsymbol{r}, \boldsymbol{p}) + \ldots,\ F(\boldsymbol{r}, \boldsymbol{p}) \text{ is real}\}. \tag{2.3}$$

Next we assume that at some initial stage the system is in an equilibrium state described by the distribution function $f_0(\boldsymbol{r}, \boldsymbol{p})$ and that at $t = t_0$ a perturbation is switched on. The distribution function $f(\boldsymbol{r}, \boldsymbol{p}, t)$ at any time $t > t_0$ is related to $f_0(\boldsymbol{r}, \boldsymbol{p})$ by a canonical transformation and therefore belongs to the set Λ. If $f_0(\boldsymbol{r}, \boldsymbol{p})$ is just a function of h_0 we may write, instead of (2.3),

$$f(\boldsymbol{r}, \boldsymbol{p}) = f_0(\boldsymbol{r}, \boldsymbol{p}) + \frac{\mathrm{d}f_0}{\mathrm{d}h_0}\{h_0, F\}(\boldsymbol{r}, \boldsymbol{p}) + \frac{1}{2}\frac{\mathrm{d}^2 f_0}{\mathrm{d}h_0^2}\{h_0, F\}^2(\boldsymbol{r}, \boldsymbol{p})$$

$$+ \frac{1}{2}\frac{\mathrm{d}f_0}{\mathrm{d}h_0}\{\{h_0, F\}, F\}(\boldsymbol{r}, \boldsymbol{p}) + \ldots \tag{2.4}$$

where $F(\boldsymbol{r}, \boldsymbol{p})$ is a generating function for a canonical transformation.

It is convenient to introduce the set of the generating functions of the normal modes of excitation of the system, $\{F_\alpha(\boldsymbol{r}, \boldsymbol{p}, t)\}$, which are defined in the following way:

$$\{F_\alpha, h_0\} = \mathrm{i}\omega_\alpha F_\alpha \qquad F_\alpha(\boldsymbol{r}, \boldsymbol{p}, t) = F_\alpha(\boldsymbol{r}, \boldsymbol{p}) \exp(-\mathrm{i}\omega_\alpha t). \tag{2.5}$$

These functions can be chosen to obey orthogonality relations

$$-\mathrm{i}\int \mathrm{d}\Gamma\{F_\alpha, F_\beta^*\} f_0 = \mathcal{N}_\alpha^2 \delta_{\alpha\beta} \tag{2.6}$$

where $\mathrm{d}\Gamma = g\, \mathrm{d}^3 r\, \mathrm{d}^3 p/(2\pi)^3$ is the volume element of phase space ($g = 4$ is the spin-isospin multiplicity) and $\mathcal{N}_\alpha$ is the normalisation constant:

$$\mathcal{N}_\alpha^2 = -\omega_\alpha \int \mathrm{d}\Gamma \frac{\mathrm{d}f_0}{\mathrm{d}h_0} F_\alpha F_\alpha^*. \tag{2.7}$$

The normalised generating functions are

$$\mathcal{F}_\alpha = F_\alpha / \mathcal{N}_\alpha.$$

Later on in this section we have to assume that the functions $\{\mathcal{F}_\alpha\}$ form a complete set. It is useful to express the transition amplitudes into the normal modes of excitation due to an external field $R(\boldsymbol{r}, \boldsymbol{p})$ in terms of these generators:

$$R_{\alpha 0} = -\mathrm{i}\int \mathrm{d}\Gamma\, f_0\{R, \mathcal{F}_\alpha^*\}. \tag{2.8}$$

In order to study small deviations from equilibrium we suppose that the generator $F(\boldsymbol{r}, \boldsymbol{p}, t)$ in expression (2.4) is infinitesimal and, therefore, we retain only the first-order terms in this expression

$$f(\boldsymbol{r}, \boldsymbol{p}, t) = f_0(\boldsymbol{r}, \boldsymbol{p}) + \delta f(\boldsymbol{r}, \boldsymbol{p}, t) \tag{2.9a}$$

where

$$\delta f(\boldsymbol{r}, \boldsymbol{p}, t) = \frac{\mathrm{d}f_0}{\mathrm{d}h_0}\{h_0, F\}. \tag{2.9b}$$

We consider our system to be perturbed by an external field $\beta(t)R(\boldsymbol{r}, \boldsymbol{p})$ and look for the solutions of the linearised Vlasov equation

$$\frac{\partial}{\partial t}\delta f + \{\delta f, h_0\} = -\beta(t)\{f_0, R\}. \tag{2.10}$$

It is useful to work with the Fourier transform of $\delta f(\boldsymbol{r}, \boldsymbol{p}, t)$

$$\delta f(\boldsymbol{r}, \boldsymbol{p}, \omega) = \int \mathrm{d}t \exp(\mathrm{i}\omega t)\delta f(\boldsymbol{r}, \boldsymbol{p}, t)$$

(we assume that ω has a small positive imaginary part $\mathrm{i}\xi$ to ensure the convergence of the integral) and to expand $\delta f(\boldsymbol{r}, \boldsymbol{p}, \omega)$ and the perturbing field $R(\boldsymbol{r}, \boldsymbol{p})$ in terms of the functions $\{\mathscr{F}_\alpha\}$

$$\delta f(\boldsymbol{r}, \boldsymbol{p}, \omega) = \sum_\alpha c_\alpha \{f_0, \mathscr{F}_\alpha\} \tag{2.11}$$

$$R(\boldsymbol{r}, \boldsymbol{p}) = \sum_\alpha R_\alpha \mathscr{F}_\alpha. \tag{2.12}$$

The coefficients of these expansions, c_α and R_α, are related by the linearised Vlasov equation

$$c_\alpha = -\mathrm{i}\beta(\omega)\frac{R_\alpha}{\omega - \omega_\alpha + \mathrm{i}\xi}. \tag{2.13}$$

From the orthogonality relations we obtain the inverse of (2.12)

$$R_\alpha = \mathrm{i}\int \mathrm{d}\Gamma\{\mathscr{F}_\alpha^*, R\} f_0. \tag{2.14}$$

The density fluctuation $\delta\rho(\boldsymbol{r}, t)$ and the current density $\boldsymbol{j}(\boldsymbol{r}, t)$ which are produced by the perturbation are defined in the usual way:

$$\delta\rho(\boldsymbol{r}, t) = \int g\frac{\mathrm{d}^3 p}{(2\pi)^3}\delta f(\boldsymbol{r}, \boldsymbol{p}, t) \tag{2.15}$$

and

$$\boldsymbol{j}(\boldsymbol{r}, t) = \int g\frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{\boldsymbol{p}}{m}\delta f(\boldsymbol{r}, \boldsymbol{p}, t). \tag{2.16}$$

We shall define the linear response function and the strength function for two special cases of $R(\boldsymbol{r}, \boldsymbol{p})$:

(a) $R(\boldsymbol{r}, \boldsymbol{p}) = Q(\boldsymbol{r})$

(b) $R(\boldsymbol{r}, \boldsymbol{p}) = \boldsymbol{p}\cdot\boldsymbol{A}(\boldsymbol{r})$

from the Fourier transforms of (2.15) and (2.16)

$$\delta\rho(\boldsymbol{r}, \omega) = \sum_\alpha \int g\frac{\mathrm{d}^3 p}{(2\pi)^3} c_\alpha\{f_0, \mathscr{F}_\alpha\} \tag{2.17}$$

$$\boldsymbol{j}(\boldsymbol{r}, \omega) = \sum_\alpha \int g\frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{\boldsymbol{p}}{m} c_\alpha\{f_0, \mathscr{F}_\alpha\}. \tag{2.18}$$

For the case $R(\boldsymbol{r}, \boldsymbol{p}) = Q(\boldsymbol{r})$ we get from (2.17), (2.13) and (2.14)

$$\delta\rho = \sum_\alpha \int g\frac{\mathrm{d}^3 p}{(2\pi)^3}\{f_0, \mathscr{F}_\alpha\}\frac{\beta(\omega)}{\omega - \omega_\alpha + \mathrm{i}\xi}\int \mathrm{d}\Gamma'\{f_0, \mathscr{F}_\alpha^*\}Q(\boldsymbol{r}'). \tag{2.19}$$

By definition the linear response function $D(\boldsymbol{r}, \boldsymbol{r}', \omega)$ is given by

$$\delta\rho = \beta(\omega)\int \mathrm{d}^3 r'\, D(\boldsymbol{r}, \boldsymbol{r}', \omega)Q(\boldsymbol{r}'). \tag{2.20}$$

Comparing this expression with (2.19) we deduce

$$D(\boldsymbol{r},\boldsymbol{r}',\omega)=\sum_\alpha \int g\frac{d^3p}{(2\pi)^3}\{f_0,\mathscr{F}_\alpha\}\frac{1}{\omega-\omega_\alpha+i\xi}\int g\frac{d^3p'}{(2\pi)^3}\{f_0,\mathscr{F}_\alpha^*\}. \tag{2.21}$$

Finally we obtain for the strength function

$$S(\omega)=-\frac{1}{\pi}\,\mathrm{Im}\int d^3r\int d^3r'\,Q(\boldsymbol{r})D(\boldsymbol{r},\boldsymbol{r}',\omega)Q(\boldsymbol{r}')$$
$$=\sum_\alpha \delta(\omega-\omega_\alpha)\int d\Gamma\,\{f_0,\mathscr{F}_\alpha\}Q(\boldsymbol{r})\int d\Gamma'\,\{f_0,\mathscr{F}_\alpha^*\}Q(\boldsymbol{r}')$$

or, defining $Q_{\alpha 0}$ by (2.8),

$$S(\omega)=\sum_\alpha \delta(\omega-\omega_\alpha)|Q_{\alpha 0}|^2. \tag{2.22}$$

For the second case $R(\boldsymbol{r},\boldsymbol{p})=\boldsymbol{p}\cdot\boldsymbol{A}(\boldsymbol{r})$ the linear response function $D_{ik}(\boldsymbol{r},\boldsymbol{r}',\omega)$ is a second-rank tensor defined in the following way [6]:

$$j_i=\beta(\omega)\sum_k\int d^3r'\,D_{ik}(\boldsymbol{r},\boldsymbol{r}',\omega)A_k(\boldsymbol{r}'). \tag{2.23}$$

From (2.18), (2.13) and (2.14) we get for the current density

$$\boldsymbol{j}=\sum_\alpha\int g\frac{d^3p}{(2\pi)^3}\frac{\boldsymbol{p}}{m}\{f_0,\mathscr{F}_\alpha\}\frac{\beta(\omega)}{\omega-\omega_\alpha+i\xi}\int d\Gamma'\,\{f_0,\mathscr{F}_\alpha^*\}\boldsymbol{p}'\cdot\boldsymbol{A}(\boldsymbol{r}').$$

Therefore

$$D_{ik}(\boldsymbol{r},\boldsymbol{r}',\omega)=\sum_\alpha\frac{1}{\omega-\omega_\alpha+i\xi}\int g\frac{d^3p}{(2\pi)^3}p_i\{f_0,\mathscr{F}_\alpha\}\int g\frac{d^3p'}{(2\pi)^3}p_k'\{f_0,\mathscr{F}_\alpha^*\} \tag{2.24}$$

and this result enables us to calculate the strength function

$$S(\omega)=-\frac{1}{\pi}\,\mathrm{Im}\int d^3r\int d^3r'\sum_{ik}A_i(\boldsymbol{r})D_{ik}(\boldsymbol{r},\boldsymbol{r}',\omega)A_k(\boldsymbol{r}')$$
$$=\sum_\alpha\delta(\omega-\omega_\alpha)|(\boldsymbol{p}\cdot\boldsymbol{A})_{\alpha 0}|^2 \tag{2.25}$$

where

$$(\boldsymbol{p}\cdot\boldsymbol{A})_{\alpha 0}=i\sum_k\int d\Gamma\,\{f_0,\mathscr{F}_\alpha^*\}p_kA_k(\boldsymbol{r}).$$

It is easy to derive the sum rules S_1 and S_3:

$$S_1=\sum_{\substack{\alpha\\(\omega_\alpha>0)}}\omega_\alpha|R_{\alpha 0}|^2=-\tfrac{1}{2}\int d\Gamma\,f_0\{R,\{h_0,R\}\} \tag{2.26a}$$

$$S_3=\sum_{\substack{\alpha\\(\omega_\alpha>0)}}\omega_\alpha^3|R_{\alpha 0}|^2=-\tfrac{1}{2}\int d\Gamma\,f_0\{\{R,h_0\},\{h_0,\{R,h_0\}\}\}. \tag{2.26b}$$

The proof is as follows:

$$-\tfrac{1}{2}\int \mathrm{d}\Gamma\, f_0\{R,\{h_0,R\}\} = -\tfrac{1}{2}\int \mathrm{d}\Gamma\, \frac{\mathrm{d}f_0}{\mathrm{d}h_0}\{h_0,R\}^2$$

$$= \tfrac{1}{2}\sum_{\alpha\beta} R_\alpha R_\beta \omega_\alpha \omega_\beta \int \mathrm{d}\Gamma\, \frac{\mathrm{d}f_0}{\mathrm{d}h_0}\mathscr{F}_\alpha \mathscr{F}_\beta = \sum_{\substack{\alpha \\ (\omega_\alpha>0)}} \omega_\alpha R_\alpha R_\alpha^*$$

$$-\tfrac{1}{2}\int \mathrm{d}\Gamma\, f_0\{\{R,h_0\},\{h_0,\{R,h_0\}\}\} = -\tfrac{1}{2}\int \mathrm{d}\Gamma\, \frac{\mathrm{d}f_0}{\mathrm{d}h_0}\{h_0,\{R,h_0\}\}^2$$

$$= -\tfrac{1}{2}\sum_{\alpha\beta} R_\alpha R_\beta \omega_\alpha^2 \omega_\beta^2 \int \mathrm{d}\Gamma\, \frac{\mathrm{d}f_0}{\mathrm{d}h_0}\mathscr{F}_\alpha \mathscr{F}_\beta$$

$$= \sum_{\substack{\alpha \\ (\omega_\alpha>0)}} \omega_\alpha^3 R_\alpha R_\alpha^*.$$

In the same way, the sum rule S_{-1} is preserved. If we consider $\beta R(\boldsymbol{r},\boldsymbol{p})$ to be a static external field we obtain

$$\sum_\alpha \frac{|R_{\alpha 0}|^2}{\omega_\alpha} = \frac{1}{2}\frac{\partial^2}{\partial\beta^2}\int \mathrm{d}\Gamma\, f(\boldsymbol{r},\boldsymbol{p})(h_0+\beta R(\boldsymbol{r},\boldsymbol{p}))|_{\beta=0}.$$

Indeed

$$\frac{1}{2}\frac{\partial^2}{\partial\beta^2}\int \mathrm{d}\Gamma\, f(\boldsymbol{r},\boldsymbol{p})(h_0+\beta R(\boldsymbol{r},\boldsymbol{p})) = \frac{1}{2}\frac{\partial^2}{\partial\beta^2}\int \mathrm{d}\Gamma\, \delta f(\boldsymbol{r},\boldsymbol{p})\beta R(\boldsymbol{r},\boldsymbol{p})$$

$$= \frac{1}{2}\frac{\partial^2}{\partial\beta^2}\sum_\alpha \beta \int \mathrm{d}\Gamma\, c_\alpha\{f_0,\mathscr{F}_\alpha\}R$$

$$= \sum_\alpha \frac{R_\alpha}{\omega_\alpha}\left(\mathrm{i}\int \mathrm{d}\Gamma\, f_0\{R,\mathscr{F}_\alpha\}\right) = \sum_\alpha \frac{|R_\alpha|^2}{\omega_\alpha}.$$

3. Harmonic oscillator static self-consistent potential

We apply this formalism to the case

$$U(r) = \tfrac{1}{2}m\Omega^2 r^2. \tag{3.1}$$

The generators of the normal modes of excitation, F_α (not normalised), are expressed in terms of the elementary functions η_i and η_i^*:

$$\eta_i = (2m\Omega)^{-1/2}(p_i + \mathrm{i}m\Omega r_i) \tag{3.2a}$$

$$\eta_i^* = (2m\Omega)^{-1/2}(p_i - \mathrm{i}m\Omega r_i) \qquad i = x, y, z. \tag{3.2b}$$

The most general form of F_α is

$$F_\alpha = \prod_i \eta_i^{a_i}\eta_i^{*b_i} \tag{3.3}$$

corresponding to the normal-mode frequency

$$\omega_\alpha = \sum_i (a_i - b_i)\Omega \tag{3.4}$$

so that F_α and F_α^* obey the equations

$$\{h_0, F_\alpha\} = \mathrm{i}\omega_\alpha F_\alpha \tag{3.5a}$$

$$\{h_0, F_\alpha^*\} = -\mathrm{i}\omega_\alpha F_\alpha^*. \tag{3.5b}$$

If in the expression (3.3) $a_i = b_i$, the generator F_α is a constant of motion and, therefore, produces no excitation

$$\left\{h_0, \prod_i \eta_i^{a_i} \eta_i^{*a_i}\right\} = 0.$$

In the last part of this work we study the response to the perturbations

$$\text{(a)} \qquad Q(\boldsymbol{r}) = r^2 Y_{20} = \frac{1}{4}\left(\frac{5}{\pi}\right)^{1/2}(2z^2 - x^2 - y^2) \tag{3.6}$$

which induces transitions in the system of the type $l^\pi = 2^+$ and will give rise to an irrotational and incompressible flow and

$$\text{(b)} \qquad \boldsymbol{p} \cdot \boldsymbol{A}(\boldsymbol{r}) = \boldsymbol{p} \cdot \frac{1}{2}\left(\frac{15}{2\pi}\right)^{1/2} z(y\hat{\boldsymbol{i}} - x\hat{\boldsymbol{j}}) \tag{3.7}$$

whose effect shall only be seen in the current density and will induce the so-called twist mode a $l^\pi = 2^-$ transition [7].

3.1. *The electric excitation*

In terms of the elementary functions η_i and η_i^*, $Q(\boldsymbol{r})$ has the form

$$Q(\boldsymbol{r}) = -\frac{1}{8m\Omega}\left(\frac{5}{\pi}\right)^{1/2}[2(\eta_z - \eta_z^*)^2 - (\eta_x - \eta_x^*)^2 - (\eta_y - \eta_y^*)^2]. \tag{3.8}$$

This operator excites normal modes with frequency $2\Omega(F_\alpha = \eta_x^2, \eta_y^2, \eta_z^2)$ and with frequency zero $(F_\alpha = \eta_x\eta_x^*, \eta_y\eta_y^*, \eta_z\eta_z^*)$.

We suppose that the equilibrium distribution function is given by the Thomas–Fermi approximation

$$f_0(\boldsymbol{r}, \boldsymbol{p}) = \theta(E_\mathrm{F} - h_0)$$

with

$$\theta(x) = \begin{cases} 1 & \text{if } x \geqslant 0 \\ 0 & \text{if } x < 0 \end{cases}$$

and E_F the Fermi energy.

From expressions (2.8) and (2.26*a*) we obtain the transition amplitudes Q_{i0} and the sum rule S_1:

$$Q_{x0} = Q_{y0} = \frac{1}{8m\Omega}\left(\frac{5gE_\mathrm{F}^4}{6\pi\Omega^4}\right)^{1/2} \qquad Q_{z0} = -\frac{1}{4m\Omega}\left(\frac{5gE_\mathrm{F}^4}{6\pi\Omega^4}\right)^{1/2}$$

and

$$S_1 = g\frac{5E_\mathrm{F}^4}{32\pi m^2\Omega^5}.$$

The transition density $\delta\rho$ and current density $\boldsymbol{j}$ caused by the field $Q(\boldsymbol{r})$ are calculated from their definitions (2.15) and (2.16) (or (2.17) and (2.18) if we are only interested in their Fourier transforms)

$$\delta\rho(\boldsymbol{r},\omega)=-\beta(\omega)\frac{\Omega}{2\pi^2}\left(\frac{1}{\omega-2\Omega+\mathrm{i}\xi}-\frac{1}{\omega+2\Omega+\mathrm{i}\xi}\right)(2mE_{\mathrm{F}}-m^2\Omega^2r^2)^{1/2}Q(\boldsymbol{r})$$

$$\boldsymbol{j}(\boldsymbol{r},\omega)=-\frac{\mathrm{i}\beta(\omega)}{6\pi^2m}\left(\frac{1}{\omega-2\Omega+\mathrm{i}\xi}+\frac{1}{\omega+2\Omega+\mathrm{i}\xi}\right)(2mE_{\mathrm{F}}-m^2\Omega^2r^2)^{1/2}\nabla Q(\boldsymbol{r})$$

and we prove that the current is proportional to the gradient of the imposed field.

3.2. The magnetic excitation

The perturbation (3.7) in terms of η_i and η_i^* is given by

$$\boldsymbol{p}\cdot\boldsymbol{A}(\boldsymbol{r})=-\frac{1}{2}\left(\frac{15}{2\pi}\right)^{1/2}(2m\Omega)^{-1/2}(\eta_z-\eta_z^*)(\eta_x^*\eta_y-\eta_x\eta_y^*)$$

and excites modes with frequency Ω corresponding to the generators $F_1=\eta_z\eta_y\eta_x^*$ and $F_2=\eta_z\eta_x\eta_y^*$. The transition amplitudes from the ground state to these excited states obtained from (2.8) are

$$(\boldsymbol{p}\cdot\boldsymbol{A})_{10}=(\boldsymbol{p}\cdot\boldsymbol{A})_{20}=-\frac{1}{4}\left(\frac{gE_{\mathrm{F}}^5}{8\pi m\Omega^6}\right)^{1/2}$$

and the energy weighted sum rule S_1 as calculated from (2.26*a*) gives

$$S_1=\sum_\alpha\Omega|(\boldsymbol{p}\cdot\boldsymbol{A})_{\alpha0}|^2=\frac{gE_{\mathrm{F}}^5}{64\pi m\Omega^5}.$$

Because of the spherical symmetry of the problem this perturbation does not give rise to any change in the density, i.e. $\delta\rho=0$. Nevertheless it induces a current density

$$\boldsymbol{j}(\boldsymbol{r},\omega)=-\mathrm{i}\beta(\omega)\frac{\Omega}{2\pi^2m}\left(\frac{1}{\omega-\Omega+\mathrm{i}\xi}+\frac{1}{\omega+\Omega+\mathrm{i}\xi}\right)(2mE_{\mathrm{F}}-m^2\Omega^2r^2)^{3/2}\boldsymbol{A}(\boldsymbol{r}).$$

Taking into account the form (3.7) of the field $\boldsymbol{A}(\boldsymbol{r})$ this mode describes a rotation of the upper half of the system in opposite phase to the lower half.

4. Generalisation of the formalism

The possibility of finding a set of generating functions $F_\alpha(\boldsymbol{r},\boldsymbol{p})$ labelled only by a discrete parameter n has been considered. However, in general, the introduction of continuous parameters γ_i may be necessary. These parameters are associated with constants of motion such as the total energy or angular momentum of the single particle. It is therefore necessary to modify the formalism proposed in this paper so that it also contains these more general cases.

Let $\{F_{n\gamma}(\boldsymbol{r},\boldsymbol{p})\}$ be a complete set of generating functions labelled by a discrete parameter n and a continuous parameter γ of the form

$$F_{n\gamma}(\boldsymbol{r},\boldsymbol{p})=\mathbb{F}_{n\gamma}(\boldsymbol{r},\boldsymbol{p})\delta(\gamma-c(\boldsymbol{r},\boldsymbol{p}))\tag{4.1}$$

where $c(\boldsymbol{r}, \boldsymbol{p})$ is a constant of motion. Instead of (2.5) these functions will satisfy the equations

$$\{F_{n\gamma}(\boldsymbol{r}, \boldsymbol{p}), h_0\} = \mathrm{i}\omega_{n\gamma} F_{n\gamma}(\boldsymbol{r}, \boldsymbol{p}) \tag{4.2}$$

where the excitation energies $\omega_{n\gamma}$ are also labelled by the continuous parameter γ. The orthogonality relations now have the form

$$-\mathrm{i} \int \mathrm{d}\Gamma\, f_0\{F_{n\gamma}, F^*_{n'\gamma'}\} = \delta_{nn'}\delta(\gamma - \gamma')\mathcal{N}^2_{n\gamma}. \tag{4.3}$$

Finally the expansion of the transition operator $R(\boldsymbol{r}, \boldsymbol{p})$, the transition density $\delta\rho(\boldsymbol{r}, t)$ or the transition current density $\boldsymbol{j}(\boldsymbol{r}, t)$ in terms of these functions will carry not only a summation over the discrete parameter n but also an integral over the continuous parameter γ, e.g.

$$R(\boldsymbol{r}, \boldsymbol{p}) = \sum_n \int \mathrm{d}\gamma\, R_{n\gamma} F_{n\gamma}(\boldsymbol{r}, \boldsymbol{p}) / \mathcal{N}_{n\gamma} \tag{4.4a}$$

$$\delta\rho(\boldsymbol{r}, \omega) = \sum_n \int \mathrm{d}\gamma \int g \frac{\mathrm{d}^3 p}{(2\pi)^3} c_{n\gamma}\{f_0, F_{n\gamma}\} / \mathcal{N}_{n\gamma} \tag{4.4b}$$

and

$$\boldsymbol{j}(\boldsymbol{r}, \omega) = \sum_n \int \mathrm{d}\gamma \int g \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{\boldsymbol{p}}{m} c_{n\gamma}\{f_0, F_{n\gamma}\} / \mathcal{N}_{n\gamma}. \tag{4.4c}$$

As an example we will obtain the functions $F_{n\varepsilon}(x, p)$ for a one-dimensional system. In this case there is one constant of the motion, the energy, and it is advisable to make a change of variables from (x, p) to (x, h_0) so that we have

$$F_{n\varepsilon}(x, p) = F_{n\varepsilon}(x, h_0) = \mathbb{F}_{n\varepsilon}(x)\delta(\varepsilon - h_0) \tag{4.5}$$

where the functions $\mathbb{F}_{n\varepsilon}(x)$ are solutions of the equation

$$\{\mathbb{F}_{n\varepsilon}(x), h_0\}\delta(\varepsilon - h_0) = \mathrm{i}\omega_{n\varepsilon}\mathbb{F}_{n\varepsilon}(x)\delta(\varepsilon - h_0) \tag{4.6a}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathbb{F}_{n\varepsilon}(x))\delta(\varepsilon - h_0)\{x, h_0\} = \mathrm{i}\omega_{n\varepsilon}\mathbb{F}_{n\varepsilon}(x)\delta(\varepsilon - h_0). \tag{4.6b}$$

If $h_0 = p^2/2m + V(x)$ then $\{x, h_0\} = p/m$ and the solutions are of the form

$$\mathbb{F}^1_{n\varepsilon}(x) = \mathscr{C} \exp\left(\mathrm{i}\omega_{n\varepsilon} \int_0^x \frac{m\, \mathrm{d}x'}{p(\varepsilon, x')}\right) \tag{4.7a}$$

and

$$\mathbb{F}^2_{n\varepsilon}(x) = \mathscr{C} \exp\left(-\mathrm{i}\omega_{n\varepsilon} \int_0^x \frac{m\, \mathrm{d}x'}{p(\varepsilon, x')}\right) \tag{4.7b}$$

with

$$p(\varepsilon, x) = (2m\varepsilon - m^2\Omega^2 x^2)^{1/2}.$$

The frequencies of the normal modes are determined by the periodic conditions

$$\omega_{n\varepsilon} T(\varepsilon) = 2\pi n \tag{4.8a}$$

with

$$T(\varepsilon)=m\oint\frac{\mathrm{d}x}{p(\varepsilon,x)} \tag{4.8b}$$

where the integral is taken over a complete cycle or if $p(\varepsilon, x)=0$ at x_1 and x_2

$$T(\varepsilon)=2m\int_{x_1}^{x_2}\frac{\mathrm{d}x}{p(\varepsilon,x)}.$$

We may also notice that if we introduce (4.7) into (4.6a), $\mathbb{F}^1_{n\varepsilon}(x)$ excites the mode with frequency $\omega_{n\varepsilon}$ and $\mathbb{F}^2_{n\varepsilon}(x)$ excites the mode with frequency $-\omega_{n\varepsilon}$ and, therefore, should be interpreted as being canonically conjugated functions. In general $\mathbb{F}^1_{n\varepsilon}(x)$ and $\mathbb{F}^2_{n'\varepsilon'}(x)$ will obey the orthogonality relations (4.3).

For the case $U(x)=\frac{1}{2}m\Omega^2x^2$ we recover (3.2) from (4.7), and $\omega_{n\varepsilon}=n\Omega$ from (4.8). When we perform the integral in the exponent of (4.7) we obtain, for $n=1$,

$$\mathbb{F}^1_{n\varepsilon}(x)=\mathscr{C}\exp\left(\mathrm{i}\sin^{-1}\frac{m\Omega x}{(2\varepsilon m)^{1/2}}\right)$$

$$=\mathscr{C}(2\varepsilon m)^{-1/2}[(2\varepsilon m-m^2\Omega^2x^2)^{1/2}+\mathrm{i}m\Omega x]$$

and

$$\mathbb{F}^2_{n\varepsilon}(x)=\mathscr{C}\exp\left(-\mathrm{i}\sin^{-1}\frac{m\Omega x}{(2m\varepsilon)^{1/2}}\right)=\mathscr{C}(2m\varepsilon)^{-1/2}[(2m\varepsilon-m^2\Omega^2x^2)^{1/2}-\mathrm{i}m\Omega x]$$

which are just the functions η and η^* defined in (3.2).

5. Conclusions

We have proposed a mathematical formalism for solving the Vlasov equation with no interaction potential. The relevant quantities for the description of the system such as the transition density and current density or the response function were expressed in terms of generating functions for the normal modes of the system. This formalism was applied to the case of a harmonic oscillator static self-consistent potential and we have obtained the response to a $I^\pi=2^+$ divergence-free irrotational field and to a non-normal parity excitation $I^\pi=2^-$. These would excite the $I^\pi=2^-$ twist mode, a $1\hbar\Omega$ isoscalar transition.

Finally the formalism was extended so that it could be applied to a more general self-consistent potential. In this case, the generating functions of the elementary excitations need to be parametrised by both discrete and continuous parameters.

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References

[1] Brink D M and di Toro M 1981 *Nucl. Phys.* A **372** 131
[2] Brink D M, Dellafiore A and di Toro M *Preprint 8/86* Oxford University
[3] da Providência J, Brito L P and da Providência C 1985 *Nuovo Cimento* A **87** 248
[4] Brito L and Providência C 1985 *Phys. Rev.* C **32** 2049
[5] di Toro M, Lombardo U and Russo G 1985 *Nuovo Cimento* A **87** 174
[6] Pines D and Nozières P 1966 *The Theory of Quantum Liquids* (New York: Benjamin)
[7] Holzwarth G and Eckart G 1977 *Z. Phys.* A **283** 219